## Abstracts:

### On the stability of the 3-body problem

### Holger Dullin (University of Sydney)

In 1998, at the ICM in Berlin, Michael Herman formulated what he called the "oldest problem in dynamics": Is the set of unbounded orbits in the 3-body problem dense for negative energies? I will explain the background of this question, and then present recent results obtained jointly with Albouy and Scheurle that show that the answer to this question is "No." The catch is that our results hold for the 3-body problem in 4-dimensional space, but do not transfer to the usual setting in 2- or 3-dimensional space. The main observation is that for 3 bodies in dimension 4 there are simple periodic solutions (relative equilibria) at which the symmetry reduced Hamiltonian has a minimum. The ultimate reason for this is that the structure of the rotational symmetry group $SO(d)$ is quite different for $d = 3$ and $d = 4$.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

### Varieties and analytic normalizations of partly integrable systems near a periodic orbit

### Xiang Zhang (Shanghai Jiaotong University)

For an analytic autonomous differential system in $\mathbb{R}^n$ with a periodic orbit, we obtain the varieties of its partly integrability near the periodic orbit and provide the asymptotic expressions of the functionally independent first integrals. We also prove that a partly analytic integrable differential system is analytically equivalent to its Poincaré-Dulac normal form on invariant manifold. Finally we show that a partly analytic integrable differential system has a $d+1$ dimensional submanifold in a neighborhood of the periodic orbit, which is filled up with periodic orbits, where $d$ is the maximal number of linearly independent elements in the resonant set associated to the Floquet multipliers of the periodic orbit. We also provide a new criterion on characterization of isochronous centers of planar analytic differential systems via first integrals.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

### Integrability and meromorphic solutions

### Thierry Combot (University of Burgundy)

Consider a rationally integrable Hamiltonian $H$, with its rational first integrals $I_1 = H, \dots, I_n$. For real systems, it follows from the Arnold Liouville Theorem that solutions are quasi periodic. In the complex domain however, there are typically more than $n$ complex periods. A natural generalisation of Arnold Liouville result is to consider $n$ times $t_1, \dots, t_n$ from associated symplectic vector fields to $I_1, \dots, I_n$, and look for meromorphic solutions in these times, assuming suitable coordinates. The existence of such solutions depends on the decomposition of the Albanese variety of a Liouville torus, and we will then generalize the notion of complete algebraic integrability. We will present some new examples which can be integrated in such a way, and present necessary condition for their existence.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

**Real Liouvillian extensions of partial differential fields**

**Zbigniew Hajto (Jagiellonian University)**

Galois theory for linear differential equations forms a counterpart of classical Galois theory. The idea of Galois of characterising those polynomial equations solvable by radicals using the group of permutations of the roots which preserve the relations between them is paralleled in the Picard-Vessiot theory by characterising solvability by quadratures employing the group of linear automorphisms of the space of solutions that preserve the differential relations between them. A satisfactory Galois theory for linear differential equations defined over a differential field $K$ of characteristic zero with algebraically closed subfield of constants $C$ was established by Kolchin under the name Picard-Vessiot theory of homogeneous linear differential equations [3]. Later, it took more than 50 years to elaborate the Picard-Vessiot theory of real and p-adic differential fields [1, 2]. In my talk, I will present Galois theory for partial differential systems defined over formally real differential fields with a real closed field of constants and over formally $p$-adic differential fields with a $p$-adically closed field of constants. For an integrable partial differential system defined over such a field, there exists a formally real (resp. formally $p$-adic) Picard-Vessiot extension. Moreover, I will comment on the uniqueness result for these Picard-Vessiot extensions and on the Galois correspondence theorem in this setting. I will explain the application of this theory to characterise formally real Liouvillian extensions of real partial differential fields with a real closed field of constants by means of split solvable linear algebraic groups. For example,in this context, Nash functions will be characterised by finite algebraic extensions. Finally, I will discuss the topological properties of real Liouville functions and relate them with the concept of tame topology in the sense of Grothendieck and Khovanskii. Some applications in the theory of real analytic dynamical systems will be presented as well.

# References

[1] T. Crespo, Z. Hajto and M. van der Put, *Real and p-adic Picard-Vessiot fields*, Math. Ann. 365 (2016), 93-103.

[2] T. Crespo, Z. Hajto and R. Mohseni, *Real Liouvillian Extensions of Partial Differential Fields*, SIGMA 17 (2021), 095, 16 pages.

[3] E. Kolchin, *Algebraic matric groups and the Picard-Vessiot theory of homogeneous lineal ordinary differential equations*, Annals of Mathematics 49 (1948), 87–128.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

**Burchnall-Chaundy polynomials for matrix ODOs and Picard-Vessiot Theory**

**Emma Previato (Boston University), Sonia L. Rueda (Universidad Politecnica de Madrid) and Maria-Angeles Zurro (Universidad Autonóma de Madrid)**

Burchnall and Chaundy showed that if two ordinary differential operators (ODOs) $P, Q$ with analytic coefficients commute then there exists a polynomial $f(\lambda, \mu)$ with complex coefficients such that $f(P, Q) = 0$, called the BC-polynomial. This polynomial can be computed using the differential resultant for ODOs. In this work we extend this result to ma- trix ordinary differential operators, MODOs. Our matrices have entries in a differential field $K$, whose field of constants $C$ is algebraically closed and of zero characteristic. We restrict to the case of order one operators $P$, with invertible leading coefficient. We define a new differential elim- ination tool, the matrix differential resultant. We use it to compute the BC-polynomial f of a pair of commuting MODOs, and we also prove that it has constant coefficients. This resultant provides the necessary and suf- ficient condition for the spectral problem $PY = \lambda Y$, $QY = \mu Y$ to have a solution. Techniques from differential algebra and Picard-Vessiot theory allow us to describe explicitly isomorphisms between commutative rings of MODOs $C[P, Q]$ and a finite product of rings of irreducible algebraic curves.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

**Nonintegrability of nearly integrable dynamical systems near regular level sets**

**Shoya Motonaga (Ritsumeikan University)**

We study the existence of real-analytic first integrals and real-analytic integrability for perturbations of integrable systems including non-Hamiltonian ones in the sense of Bogoyavlenskij such that the first integrals and commutative vector fields depend analytically on the small parameter. We compare our results with the classical results of Poincaré and Kozlov for systems written in action and angle coordinates and discuss their relationships with the subharmonic and homoclinic Melnikov methods for periodic perturbations of single-degree-of-freedom Hamiltonian systems. Moreover, we apply our theory to the periodically forced Duffing oscillator and reveal that the perturbed systems can be real-analytically nonintgrable even if there exists no transverse homoclinic orbit to a periodic orbit.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

**Non-integrability criterion for homogeneous Hamiltonian systems via blowing-up theory**

**Mitsuru Shibayama (Kyoto University)**

Distinguishing between integrable and nonintegrable Hamiltonian systems poses a significant challenge. In this study, we propose a novel approach to establish the non-integrability of homogeneous Hamiltonian systems with two degrees of freedom. The homogeneity degree can encompass real values, not limited to integers. Our proof is grounded in the blowing-up theory, originally introduced by McGehee in the context of the collinear three-body problem. Furthermore, we compare our findings with the Molares-Ramis theory, known as the most robust theory in this field.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

**Nonintegrability of the restricted three-body problem**

**Kazuyuki Yagasaki (Kyoto University)**

The problem of nonintegrability of the circular restricted three-body problem is very classical and important in the theory of dynamical systems. It was partially solved by Poincaré in the nineteenth century: He showed that there exists no real-analytic first integral which depends analytically on the mass ratio of the second body to the first one and is functionally independent of the Hamiltonian. When the mass of the second body becomes zero, the restricted three-body problem reduces to the two-body Kepler problem. We prove the nonintegrability of the restricted three-body problem both in the planar and spatial cases for any nonzero mass of the second body. Our basic tool of the proofs is a technique developed here for determining whether perturbations of integrable systems which may be non-Hamiltonian are not meromorphically integrable near resonant periodic orbits such that the first integrals and commutative vector fields also depend meromorphically on the perturbation parameter. The technique is based on generalized versions due to Ayoul and Zung of the Morales-Ramis and Morales-Ramis-Simó theories.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

**Semiclassical quantification of some two degree of freedom potentials: a differential Galois approach**

**Juan José Morales-Ruiz (Universidad Politecnica de Madrid)**

In this work we explain the relevance of the Differential Galois Theory in the semiclassical (or WKB) quantification of some two degree of freedom potentials. The key point is that the semiclassical path integral quantification around a particular solution depends on the variational equation around that solution: a very well-known object in dynamical systems and variational calculus. Then, as the variational equation is a linear ordinary differential system, it is possible to apply the differential Galois theory to study its solvability in closed form. We obtain closed form solutions for the semiclassical quantum fluctuations around constant velocity solutions for some systems like the classical Hermite/Verhulst, Bessel, Legendre, and Lame potentials. We remark that some of the systems studied are not integrable, in the Liouville - Arnold sense. This work can be considered as first step of a research program started by the spiker with some collaborators about a differential Galois approach to quantum physics. If time permits we will also point out some insights about that program. (joint work with P. Acosta-Humánez, J.T. Lázaro and Ch. Pantazi)

REFERENCE:

P. B. Acosta-Humánez, J. T. Lázaro, J. J. Morales-Ruiz, Ch. Pantazi, Semiclassical quantification of some two degree of freedom potentials: a differential Galois approach. arXiv:2307.09318 .

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

**On some collinear configurations in the planar three-body problem**

**Alexei Tsygvintsev (École Normale Supérieure de Lyon)**

We further investigate the planar Newtonian three-body problem with a focus on collinear configurations, where either the three bodies or their velocities are aligned. We provide an independent proof of Montgomery's result, stating that apart from the Lagrange's solution, all negative energy solutions to the zero angular ¥ momentum case result in syzygies, i.e., collinear configurations of positions. The concept of generalised syzygies, inclusive of velocity alignments, was previously explored by the author for bounded solutions. In this study, we broaden our scope to encompass negative energy cases and provide new bounds. Our methodology builds upon the elementary Sturm-Liouville theory and the Wintner-Conley “ linear ” form of the three-body problem, as previously explored in the works of Albouy and Chenciner.